Exercise 2.2.5

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

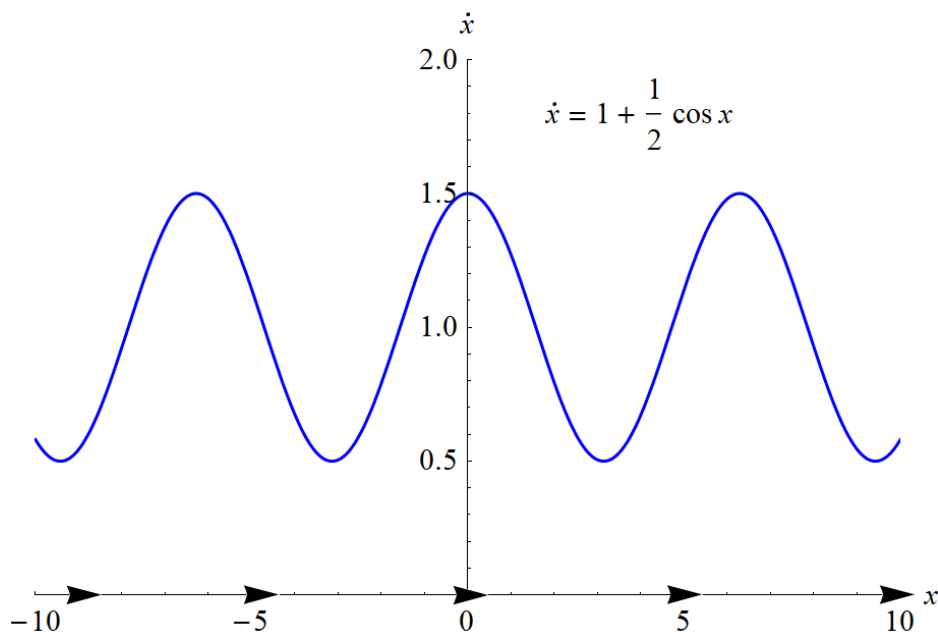$$\dot{x} = 1 + \frac{1}{2}\cos x$$

Solution

The fixed points of the flow occur where $\dot{x} = 0$.

$$1 + \frac{1}{2}\cos x^* = 0$$

$$\cos x^* = -2$$

$$x^* = \{\}$$

There are no fixed points. This is illustrated below in the graph of $\dot{x}$ versus x.

When the function is negative the flow moves to the left, and when the function is positive the flow moves to the right.

The aim now is to solve the following initial value problem.

$$\frac{dx}{dt} = 1 + \frac{1}{2}\cos x, \quad x(0) = x_0$$

Solve the ODE by separating variables and integrating both sides.

$$\frac{dx}{1 + \frac{1}{2}\cos x} = dt$$

$$\int \frac{dx}{1 + \frac{1}{2}\cos x} = \int dt$$

$$\frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right) = t + C \tag{1}$$

Now apply the initial condition to determine C.

$$\frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right) = C$$

As a result, equation (1) becomes

$$\frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right) = t + \frac{4}{\sqrt{3}} \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right)$$

$$\tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x}{2}\right) = \frac{\sqrt{3}}{4} t + \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right)$$

$$\frac{1}{\sqrt{3}} \tan \frac{x}{2} = \tan\left[\frac{\sqrt{3}}{4} t + \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right)\right]$$

$$\tan \frac{x}{2} = \sqrt{3} \tan\left[\frac{\sqrt{3}}{4} t + \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right)\right]$$

$$\frac{x}{2} = \tan^{-1}\left\{\sqrt{3} \tan\left[\frac{\sqrt{3}}{4} t + \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right)\right]\right\} + n\pi.$$

Therefore,

$$x(t) = 2\tan^{-1}\left\{\sqrt{3} \tan\left[\frac{\sqrt{3}}{4} t + \tan^{-1}\left(\frac{1}{\sqrt{3}} \tan \frac{x_0}{2}\right)\right]\right\} + 2n\pi,$$

where n is an integer. n should be chosen on different intervals of t so that the graph of $x(t)$ versus t is continuous.

A qualitative sketch of x versus t is shown for various initial conditions.